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SOME NEW DYNAMICAL EFFECTS IN THE PERTURBED EULER-POINSOT PROBLEM, DUE TO SPLITTING OF SEPARATRICES*

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An investigation is presented of a series of new qualitative dynamical effects, due to the phenomenon of the splitting of the asymptotic surfaces (separatrices) of perturbed permanent rotations in the motion of an asymmetric rigid body with fixed point in a weak gravitational field (or, in greater generality, an axisymmetric irrotational field). A quantitative index of the non-coincidence of the separatrices is defined and appropriate estimates are established. Conditions are found which, when imposed on the parameters of the problem, imply the existence of invariant tori separating perturbed hyperbolic permanent rotations. It is shown that for almost all values of the parameters there exist quasirandom motions due to the existence of transversally intersecting separatrices. Bifurcation effects, represented by infinitely many changes in the qualitative behaviour pattern of the trajectories as the Poincaré parameter tends to zero, are observed and studied. This paper is a continuation of /1/.

1. *Splitting of separatrices and the method of normal Moser coordinates.*

Let U be a domain in the real plane $\mathbf{R}^2\{x^1, x^2\}$, and μ a small parameter, $|\mu| < \varepsilon$. We consider the system

$$\frac{dx^1}{d\varphi} = \frac{\partial H}{\partial x^2}, \quad \frac{dx^2}{d\varphi} = -\frac{\partial H}{\partial x^1} \tag{1.1}$$

$$(H(x^1, x^2, \varphi, \mu) = H_0(x^1, x^2) + \mu H_1(x^1, x^2, \varphi) + \ldots)$$

where the Hamiltonian is a 2π-periodic function of the time φ, analytic on the direct product

$$U\{x^1, x^2\} \times S^1\{\varphi \bmod 2\pi\} \times (-\varepsilon, \varepsilon)$$

Let the unperturbed Hamiltonian system

$$\frac{dx^1}{d\varphi} = \frac{\partial H_0}{\partial x^2}, \quad \frac{dx^2}{d\varphi} = -\frac{\partial H_0}{\partial x^1} \tag{1.2}$$

have fixed hyperbolic points $x_1, x_2, x_3 \in U$ (x_1 and x_3 may coincide), connected by two doubly asymptotic solutions $x_1^*(\varphi), x_2^*(\varphi)$ that lie in the interior of U:

$$\lim_{\varphi\to-\infty} x_k^*(\varphi) = x_k, \quad \lim_{\varphi\to+\infty} x_k^*(\varphi) = x_{k+1}; \quad k = 1, 2$$

The solutions that are asymptotic as $\varphi \to -\infty$ or $\varphi \to +\infty$ to a given periodic hyperbolic solution form two invariant surfaces, known respectively as outgoing and incoming separatrices.

System (1.2) has two pairs of coinciding (double) asymptotic surfaces of hyperbolic periodic solutions: the outgoing separatrix Γ_1'' of the solution $x \equiv x_1$ and incoming separatrix Γ_1' of the solution $x \equiv x_2$, on the one hand, and the outgoing separatrix Γ_2' of the solution $x \equiv x_2$ and incoming separatrix Γ_2'' of the solution $x \equiv x_3$, on the other.

If $\mu \neq 0$ is small, the 2π-periodic hyperbolic solutions $x \equiv x_i$ $(i = 1, 2, 3)$ and their asymptotic surfaces do not disappear, but are only somewhat deformed. In the general case, however, as observed by Poincaré, the separatrices may cease to be double (split) for small values of the parameter $\mu \neq 0$.

Suppose that for small $\mu > 0$ the solutions $x \equiv x_i$ became solutions $x = x_i(\varphi)$ and the perturbed separatrices Γ_1', Γ_1'' and Γ_2', Γ_2'' split and do not intersect, so that Γ_k'' lies to one side of Γ_k' (a section of the plane $\varphi = \text{const}$ is illustrated in Fig.1). Simple sufficient conditions have been derived /1/ for the separatrices Γ_k'' to remain distinct for

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all small $\mu > 0$ and the results have been used to study the sepatrices of the perturbed Euler-Poinsot problem. The proofs were based on the use of normal Moser coordinates (see below).

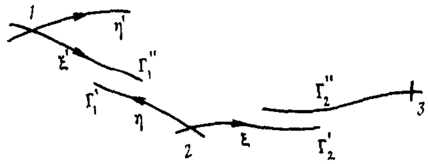

Fig.1

The "uniform version" of Moser's theorem states the following. There is a change of variables

$$x = \Phi(\xi, \eta, \varphi, \mu) = \Phi_0(\xi, \eta) + \mu\Phi_1(\xi, \eta, \varphi) + \ldots$$
$$\partial(x^1, x^2)/\partial(\xi, \eta) \equiv 1, \quad \Phi_0(0, 0) = x_2$$

which is real-analytic in ξ, η, φ, μ for sufficiently small $|\xi|, |\eta|, |\mu|$ and 2π-periodic in φ, under which system (1.1) assumes normal form (the dot denotes differentiation with respect to ω)

$$d\xi/d\varphi = \partial F/\partial\eta, \; d\eta/d\varphi = -\partial F/\partial\xi \tag{1.3}$$
$$\omega = \xi\eta, \; F(\omega, \mu) = F_0(\omega) + \mu F_1(\omega) + \ldots, \; F_0^{\cdot}(0) = \Lambda > 0$$

We may assume that the outgoing separatrix $\eta = 0, \xi > 0$ is Γ_2', and the incoming separatrix $\xi = 0, \eta > 0$ is Γ_1'. The coordinates ξ, η, φ are known as normal coordinates. Using them systematically, one can detect and investigate various new dynamical effects due to the splitting of the separatrices. The important points here are, first, that system (1.3) is trivially integrable, and, second, that one has formulae transforming from normal coordinates in the neighbourhood of the solution $x = x_i(\varphi)$ to normal coordinates in the neighbourhood of the solution $x = x_{i+1}(\varphi)$ /1/. These formulae are defined in neighbourhoods V_i of the separatrices Γ_i', Γ_i'' that do not contain the unperturbed solutions $x \equiv x_i, x \equiv x_{i+1}$, respectively. In domains of type V_1 and V_2 it is convenient to transform from ξ, η to coordinates J, φ_1 and J, φ_2, respectively, where $J = \mu^{-1}\omega$ and the phases φ_1, φ_2 are related to η, ξ by the conditions

$$x_1^*(\tau + \varphi_1) = \Phi_0(0, \eta \exp(-\Lambda\tau)) \tag{1.4}$$
$$x_2^*(\tau + \varphi_2) = \Phi_0(\xi \exp(\Lambda\tau), 0) \tag{1.5}$$

A special role in the transformation formulae is played by certain improper integrals

$$J_i(\varphi) = \int_{-\infty}^{+\infty} \{H_0, H_1\}(x_i^*(\tau - \varphi), \tau)\, d\tau$$

known as the characteristic integrals, which are 2π-periodic functions. In particular, to a first approximation with respect to μ the transformation formulae in V_i, expressed in terms of J, φ_1, φ_2 coordinates, depend only on the functions $J_i(\varphi)$ and the characteristic exponents Λ_i, Λ_{i+1} of the unperturbed hyperbolic solutions $x \equiv x_i, x \equiv x_{i+1}$. Therefore, the relative positions of the separatrices Γ_i', Γ_i'' are determined by the behaviour of the functions $J_i(\varphi)$ /2, 1/. The case illustrated in Fig.1 is possible only if $J_1(\varphi) \geqslant 0$, $J_2(\varphi) \leqslant 0$.

Henceforth, $R_1, \ldots, R_{12}$ will denote certain analytic functions and $C_1, \ldots, C_{11}$ constants.

2. Criteria for separatrices to be distinct. Order of non-coincidence.

Definition 1. We shall say that two positive variables quantities (depending on μ) are of the same order if their quotient remains bounded between two positive constants (for all small $\mu > 0$).

Definition 2 (see Fig.1). Let W_2, W_3 be small, fixed neighbourhoods of the unperturbed hyperbolic solutions $x \equiv x_2, x \equiv x_3$, whose boundaries $\partial W_2, \partial W_3$ are smooth surfaces transversally intersecting the unperturbed separatrix x_2^*. Consider the sections $\Gamma_1''^{\sim}$, $\Gamma_2''^{\sim}$ of Γ_1'', Γ_2'' lying near x_2^* outside the fixed neighbourhoods of x_2, x_3. Then the index of non-coincidence (IN) of the separatrices Γ_1'', Γ_2'' is defined as the infimum of the set of ρ such that $\Gamma_2''^{\sim}$ lies in the ρ-neighbourhood of $\Gamma_1''^{\sim}$.

We shall assume henceforth that $J_1 \not\equiv 0$ or $J_2 \not\equiv 0$, i.e., splitting of at least one of the separatrices x_1^*, x_2^* is captured by first-order perturbation theory.

Theorem 1. 1) Definition 2 may be modified in either of several ways: a) replacing x_2^* by x_1^* and the neighbourhood W_3 of the unperturbed solution $x \equiv x_3$ by an analogous neighbourhood W_1 of the solution $x \equiv x_1$; or b) taking other small neighbourhoods of the solutions $x \equiv x_2$, $x \equiv x_3$; or c) considering the set of ρ such that $\Gamma_1''^{\sim}$ lies in the ρ-neighbourhood of $\Gamma_2''^{\sim}$. In any case, the new IN of the separatrices Γ_1'', Γ_2'' is of the same order as the old one.

2) the separatrices Γ_1'', Γ_2'' remain distinct for all sufficiently small $\mu > 0$ if one of the following conditions is satisfied:

A) For no C_1 is it true that

$$J_1(\varphi) = -J_2(\varphi - C_1 - \Lambda^{-1} \ln J_1(\varphi)) \qquad (2.1)$$

B) For no C_1 is it true that

$$-J_2(\varphi) = J_1(\varphi + C_1 + \Lambda^{-1} \ln(-J_2(\varphi))) \qquad (2.2)$$

Conditions A and B are equivalent and, in particular, are satisfied if at least one of the following criteria 1°, 2°, 3° is valid:

1°. $\frac{d}{d\varphi} \ln J_1(\varphi) \geqslant \Lambda$ or $\frac{d}{d\varphi} \ln(-J_2(\varphi)) \leqslant -\Lambda$

for some φ (this is true, in particular, if $J_1(\varphi) = 0$ or $J_2(\varphi) = 0$ for some φ).

2°. The functions J_1, $-J_2$ are defined in different domains.

3°. One of the functions J_1, J_2 is not identically a constant and has either a zero or a pole on its Riemann surface; the complete analytic function /3/ corresponding to the second function is univalent (these conditions are satisfied, for example, by real trigonometric polynomials).

The equivalence of conditions A, B follows from the fact that identities (2.1), (2.2) are valid for the same values of C_1.

C) 4°. $F_0^{\cdot\cdot}(0) \neq 0$ and at least one of the functions J_i is not a constant.

3) if conditions A, B are satisfied, the IN of the separatrices Γ_1'', Γ_2'' is of order μ.

If conditions A, B fail to hold, i.e., identities (2.1) and (2.2) are valid for some C_1 but criterion 4° is satisfied, then the IN of the separatrices is bounded between two numbers of orders μ and $-\mu^2 \ln \mu$, respectively. Moreover, there exist sequences of positive $\mu \to 0$ such that the IN's are of minimal and maximal orders $-\mu^2 \ln \mu$ and μ.

This theorem is a sharper and stronger version of Theorem 1 of /1/. The rigorous proof, though elementary, is rather cumbersome. Suffice to say that it relies on ideas from /1/ and some standard theorems of analysis.

3. *Non-coincidence of the separatrices in the motion of an asymmetric rigid body in a weak axisymmetric irrotational force field.*

We consider the motion of an asymmetric rigid body about a fixed point. Let $a < b < c$ be the reciprocals of the principal moments of inertia of the body. We shall assume that the force field is weak and can be expanded in powers of a small parameter μ. Fixing some total energy level $h > 0$ and an area constant H, one can use isoenergetic reduction (also known as reduction of order, see Whittaker /4/) to transform to a reduced system of type (1.1), in which $x^1 = l$, $x^2 = L$, $\varphi = g$ are the canonical Andoyer-Deprit variables.

For $\mu = 0$, system (1.1) has fixed points

$$\gamma_1\colon (L = 0,\ l = \pi \bmod 2\pi),\ \gamma_2\colon (L = 0,\ l = 0 \bmod 2\pi)$$

connected by doubly asymptotic solutions. For $\mu = 0$, with suitable choice of parameters, the separatrices may split and do not intersect. In that case we are in the situation studied in Sects.1 and 2, but the solutions $x = x_1(\varphi)$, $x = x_3(\varphi)$ are identical. The separatrices Γ_k'' will intersect at least in two distinct homoclinic solutions, as follows from Moser's invariant curve theorem and the fact that the succession mapping of system (1.1) has an invariant surface /1/.

In our previous paper /1/ we considered motion in a gravitational force field, when the improper integrals $J_i(\varphi)$, evaluated along the unperturbed doubly asymptotic solutions, are non-constant trigonometric polynomials /2, 1/ and so Criterion 3° is effective. In the general case the integral $J_i(\varphi)$ need not be trigonometric polynomials. However, as follows from the theorem below, Criterion 4° will apply if at least one of the functions $J_i(\varphi)$ is not a constant.

Theorem 2. In the Euler-Poinsot problem one always has $F_0''(0) \neq 0$.

The proof relies on the fact that the unperturbed Hamiltonian $-G = H_0(l, L)$ — is a solution of the equation /2/

$$\tfrac{1}{2}(a \sin^2 l + b \cos^2 l)(G^2 - L^2) + \tfrac{1}{2}cL^2 = h \;\; (h = \tfrac{1}{2}bG_0^2)$$

The following theorem holds.

Theorem 3. Identities (2.1) and (2.2) cannot hold simultaneously for all four pairs of neighbouring unperturbed separatrices (even with different constants C_1), if at least one of the functions $J_i(\varphi)$ is not a constant. Thus, Conditions A and B are satisfied for some pair of separatrices.

Indeed, identities (2.1), (2.2) establish a one-to-one correspondence between the monotonicity intervals of the positive functions $J_1(\varphi)$, $-J_2(\varphi)$, such that even interval in which $J_1(\varphi)$ is an increasing (decreasing) function corresponds to a shorter (longer) interval in which $-J_2(\varphi)$ is an increasing (decreasing) function. The rest of the proof is obvious.

4. *Kolmogorov tori separating perturbed hyperbolic permanent relations.*

Consider the motion of an asymmetric rigid body in a weak gravitational force field. Denote the parameters of the problem by $\mathrm{pr} = (a, b, c, X_0, Y_0, Z_0, H/G_0)$, where X_0, Y_0, Z_0 are the direction cosines of the radius-vector of the centre of gravity relative to principal axes of inertia in the frame of the fixed point (the Poincaré parameter μ here is the product of the weight of the body and the distance between the centre of gravity and the fixed point).

Theorem 4. There exists a domain S_0 in the parameter space such that for small $\mu > 0$ there exist two-dimensional invariant tori in a neighbourhood of the unperturbed separatrices which separate perturbed periodic solutions γ_1, γ_2. Hence there exist no heteroclinic solutions.

Proof. Let $x_i^*(\varphi)$ $(i = 1, 2, 3, 4)$ denote unperturbed doubly asymptotic solutions, so chosen that the points $x_i^*(0)$ are equidistant from the fixed points γ_i (Fig.2). Let $J_i(\varphi)$ be the corresponding characteristic integrals, as calculated in /1/. We know that $(-1)^{i+1} \oint J_i(\varphi)d\varphi = 2\pi J_0$.

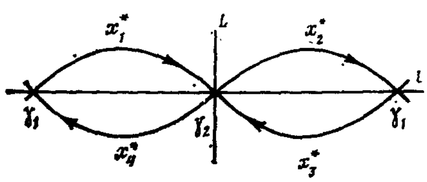

Fig.2

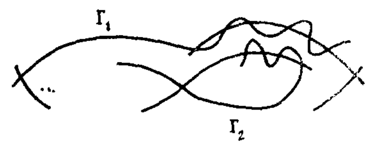

Fig.3

In the neighbourhood of the perturbed periodic solution γ_i, choose normal coordinates ξ_i, η_i, and let $\omega_i = \xi_i\eta_i$. In the neighbourhood of x_i^* we have formulae of type (1.5) for the relation between $|\xi_{i \bmod 2}|$ and the phases φ_i. Let Π_i be two-dimensional area elements defined in the neighbourhoods of x_i^* by the equations $\varphi_i = 0$. Define coordinates in each Π_i by $\varphi \bmod 2\pi$, $I_i = \mu^{-1}\omega_{i \bmod 2}$. Let $j = (i+1) \bmod 4$. To fix ideas, suppose that the perturbed separatrices are situated as shown in Fig.3. In this case $J_0 > 0$.

Lemma. The phase flow of system (1.1) carries each point $(I_i, \varphi_i = 0, \varphi)$, $(-1)^i I_i > 0$ on Π_i in time

$$\Delta\varphi = -C_2 - \Lambda^{-1}(\ln((-1)^j I_j) + \ln \mu)(1 + \mu R_i(I_j, \mu)) + \mu R_{4+i}(I_i, \varphi, \mu)$$

into the point

$$(I_j, \varphi_j = 0, \varphi + \Delta\varphi)$$
$$(I_j = I_i + \Lambda^{-1}J_i(\varphi) + \mu R_{8+i}(I_i, \varphi, \mu))$$

lying on Π_j, provided that $(-1)^j I_j > 0$.

The proof follows from Eqs.(1.3) and the transformation formulae of /1/. Thanks to the symmetry of the unperturbed problem, the constant C_2 is independent of i.

Thus, we can consider the mapping

$$(I_i, \varphi_i = 0, \varphi) \rightarrow (I_j, \varphi_j = 0, \varphi + \Delta\varphi)$$

of Π_i into Π_j; similarly, Π_j is mapped into the next area element. One can also consider a mapping T of Π_1 into itself - the composition of successive mappings of this type.

Fix constants Λ and $0 < C_3 < 1/2$. If the functions $(-1)^{i+1} J_i(\varphi)$ are δ-close to Λ in some complex neighbourhood of the real circle $S^1\{\varphi\}$ (δ-close in the C^ω-metric), then for sufficiently small $\mu > 0, \delta$ the mapping T is defined in an annulus $C_3 < -I_1 < 1 - C_3$ and is $O(\delta)$-close in the C^ω-metric to the mapping T_0:

$$I_1 \to I_1$$
$$\varphi \to \varphi - 4C_2 - 2\Lambda^{-1}(\ln(-I_1) + \ln(1 + I_1)) - 4\Lambda^{-1}\ln\mu$$

It follows from the formulae for $J_i(\varphi)$ /1/ that, by suitable choice of the parameters pr, one can guarantee the validity of these conditions. The mapping T preserves the Poincaré-Cartan integral invariant and satisfies all conditions of Moser's invariant curve theorem, provided that δ is sufficiently small and $\mu < \mu(\text{pr})$. The trajectories of system (1.1) passing through the invariant curve of T fill out the required tori.

It is essential that δ can be chosen independently of μ. A similar result will hold in the general case - the motion of a body in a weak axisymmetric irrotational force field - if one requires that the characteristic integrals differ by a small amount from non-zero constants.

It is known /5, 6/ that intersecting separatrices form a rather tangled net, which cannot intersect the invariant tori and therefore does not enable the tori to be determined near the separatrices. One can prove the following result.

Theorem 5. If all $J_i(\varphi) \not\equiv 0$ and the pairs of functions $\varepsilon J_i(\varphi), \varepsilon J_j(\varphi)$ (where $j = (i \pm 1) \bmod 4$, $\varepsilon = \mp(-1)^i$) satisfy Conditions A, B, then every invariant torus of the problem which intersects the plane $\varphi = \text{const}$ is a closed curve passing near x_k^*, x_l^* (where $l = (k + 1) \bmod 4$, $k = 1, 2, 3, 4$) or $x_1^*, \ldots, x_4^*$ will lie in a domain around γ_i defined by $|\omega_{i \bmod 2}| > C_4|\mu|$, $C_4 > 0$, provided that $\mu < \mu(\text{pr})$.

In this case the form of the mapping T_0 implies an intersecting phenomenon. The Poincaré rotation number /7/ on an invariant torus (the limit of the quotient of the increment to φ by the number of revolutions about the separatrices $x_1^*, x_2^*, x_3^*, x_4^*$) will exceed $-4(C_2 + \Lambda^{-1}\ln\mu - \Lambda^{-1}\ln 2) + O(\delta)$. Let us call the torus situated midway between splitting separatrices the "centre torus". As μ is increased, the invariant tori move from the neighbourhood of the centre torus toward the separatrices. As μ is reduced, conversely, they move from the neighbourhood of the separatrices toward the centre torus. It follows from Theorem 5 that the rotation number on each invariant torus is at most $-4(C_5 + \Lambda^{-1}\ln\mu)$. Summarizing, we see that the migration pattern of the tori as μ varies is as follows: a torus with fixed rotation number is "born" near a separatrix (centre torus), moves, and then "dies" near the centre torus (separatrix). Consequently, as μ tends to zero the invariant tori experience infinitely many birth-death bifurcations.

5. *A rigid body containing cavities filled with an ideal incompressible liquid.*

If the motion of the liquid is irrotational at some instant of time, i.e., the velocity can be expressed as the gradient of a univalent function, then by Thompson's Theorem this property is conserved at all times. The motion of the rigid body will be described by the Euler-Poisson equations, corresponding to the motion of a body whose mass equals the total mass of the body-plus-liquid system; the inertia tensor is derived from that of the original body by adding the tensor of added moments of the liquid /8/.

If all three principal moments of inertia of the system relative to the fixed point are distinct, all results of Sects.1-4 and /1/ remain valid. However, the moments of inertia of the body need not satisfy the inequality $a^{-1} < b^{-1} + c^{-1}$ /9/. It turns out that as a result the restriction $\Lambda < 1$ /1/ is relaxed and Λ may take arbitrary positive values. To verify this, it suffices to use the formula $\Lambda = b^{-1}[(b - a)(c - b)]^{1/2}$ (see /1/) and to consider the case in which the body contains an ellipsoidal liquid-filled cavity. The appropriate formulae for the added moments of inertia of the lqiuid are well-known /8, 10/.

It is proved in /1/, in particular, that there exists a domain S_3 in parameter space, satisfying the following condition. If $\text{pr} \in S_3$, then for all small $\mu > 0$ the perturbed separatrices split, do not intersect and are situated as shown in Fig.3; in addition, there exist sequences of positive numbers $\mu_n^+ \to 0, \mu_n^- \to 0, n \to \infty$, such that at $\mu = \mu_n^-$ the outgoing separatrix Γ_1 and incoming separatrix Γ_2 intersect near the unperturbed separatrices x_1^*, x_2^*, x_3^*, and at $\mu = \mu_n^+$ they do not intersect. Thanks to the fact that Λ may now be increased at will, one can prove the following result.

Theorem 6. There exists a domain $S_4 \subset S_3$ in the parameter space, satisfying the following condition. If $\text{pr} \in S_4$, there exist sequences of positive numbers $\mu_n^+ \to 0$, $\mu_n^- \to 0$, $n \to \infty$ such that at $\mu = \mu_n^-$ the separatrices Γ_1 and Γ_2 intersect near x_1^*, x_2^*, x_3^* and at $\mu = \mu_n^+$ there exist two-dimensional invariant tori situated near $x_1^*, x_2^*, x_3^*, x_4^*$ and separating the perturbed permanent rotations γ_i. Consequently, as $\mu > 0$ approaches zero one has infinitely many birth-death bifurcations of heteroclinic solutions and separating Kolmogorov tori.

Outline of proof. We first observe that the proof in /1/ can be considerably simplified. Indeed, if $X_0 = Z_0 = 0$, $Y_0 = 1$ we have

$$J_i(\varphi) = (-1)^i (a_0 - a_y \cos\varphi), \quad i = 1, 2$$
$$J_i(\varphi) = (-1)^i (a_0 + a_y \cos\varphi), \quad i = 3, 4$$

Fix $a_0 = -1$, $1/\sqrt{5} < a_y < 1$, and choose Λ to be sufficiently large $(\Lambda > \Lambda(a_y))$ (which is possible, see /1/). Then the formulae in /1/ take a rather simple form and involve small terms $O(\Lambda^{-1})$. The sequence μ_n^- will correspond to the following number t /1/:

$$(t = C_2 + \Lambda^{-1}\ln\mu + 2\pi n + \Lambda^{-1}\ln\Lambda^{-1}, \ n = [-\ln\mu/(2\pi\Lambda)])$$

satisfying the condition $t \bmod \pi = \pi/2$, and the sequence μ_n^+ to a number t such that $t \bmod \pi = 0$. It turns out that if a_y is not an element of a certain (probably empty) set, which has no limit points in the interior of $[0, 1)$, then one can take sufficiently small numbers $\mu > 0$ such that $t \bmod 2\pi = 0$ to form the required sequence μ_n^+. The proof relies on the fact that the mapping T constructed in Sect.4 is close to the identity.

We now proceed to a more rigorous discussion. Let $\mu > 0$ be sufficiently small and $t \bmod 2\pi = 0$. After a few easy manipulations, using an idea from /11/, one can show that the mapping T is an exact canonical mapping in the coordinates $s_1 = s_1(I_1) = \Lambda I_1 + O(\mu)$, $\varphi \bmod 2\pi$, which is $(O(\Lambda^{-2}) + O(\mu\ln\mu))$-close together with its first k derivatives to a translation during a time Λ^{-1} along the trajectories of the autonomous system

$$\varphi^{\cdot} = -\partial H/\partial s_1, \quad s_1^{\cdot} = \partial H/\partial\varphi \tag{5.1}$$
$$H = \sum_{i=1}^{4} (-1)^i f((-1)^i s_i), \quad f(\tau) = (\ln\tau - 1)\tau$$
$$s_{i+1} = s_1 + G_i(\varphi), \quad G_i(\varphi) = \sum_{j=1}^{i} J_j(\varphi), \quad G_4 \equiv 0$$

Here the estimate $O(\mu\ln\mu)$ depends on Λ, the estimates $O(\Lambda^{-2}), O(\mu\ln\mu)$ are uniform on any compact subset of the domain of definition $|a_y\cos\varphi| - 1 < s_1 < 0$ of H and k is any preassigned natural number. (In general, it is obviously impossible to ensure smallness in the C^ω-metric for large Λ, since the width of the corresponding complex domain depends on Λ and may tend to zero as $\Lambda \to \infty$.)

The trajectories of system (5.1) are levels of the Hamiltonian H. Using the implicit function theorem, one can show that a level γ_0: $H = H_0$ is given by an equation $s_1 = g(\varphi)$, where φ ranges over the entire circle S^1, if and only if H_0 lies in the interval $(f(1 + a_y) + f(1 - a_y); f(2a_y) - 2f(1 - a_y))$, which is non-empty for $0 \leqslant a_y < 1$. The time $T(H_0)$ necessary for a phase point $(s_1, \varphi) \in \gamma_0$ of system (5.1) to return to its initial position is not constant when $a_y \doteq 0$, and so, by analytic continuation, the same holds for all $a_y \in (0, 1)$ with the possible exception of a subset with non limit points in $[0, 1)$. Then, by Moser's invariant curve theorem, the mapping T has an invariant closed curve γ close to some level γ_0 if k, Λ are sufficiently large and $\mu < \mu(\mathrm{pr})$. The trajectories of system (1.1) passing through a curve $\gamma \subset \Pi_1$ fill out the required invariant torus, which separates perturbed hyperbolic solutions γ_1, γ_2. It remains to observe that all points in the parameter space close to those chosen in the proof also possess the necessary property.

Suppose now that some of the liquid-filled cavities are not simply connected. Instead of assuming that the liquid is in irrotational motion, let us consider a similar but rotational, eddy-free motion; then the Euler-Poisson equations will include small gyroscopic terms /8/. In that case Theorem 6 remains valid.

6. *Transversal homoclinic solutions and quasirandom motions in the dynamics of a heavy rigid body.* Under fairly general assumptions, the existence of transversally intersecting separatrices implies the existence of quasirandom motions (a theorem due to Alekseyev /12, 13/). In particular, as observed by Ziglin, quasirandom motions will appear in the perturbed Euler-Poinsot prolem if the splitting separatrices are transversally intersecting. This will happen when all the functions $J_i(\varphi)$ are of fixed signs. Supposing the contrary and omitting the Hess-Appelroth case, one obtains the model problem of Sect.1, with the hyperbolic periodic solutions $x = x_1(\varphi)$, $x = x_3(\varphi)$ coinciding and

$$J_1(\varphi) = c + \alpha_1\cos\varphi + \beta_1\sin\varphi, \quad -J_2(\varphi) = c + \alpha_2\cos\varphi + \beta_2\sin\varphi \tag{6.1}$$

where $J_1(\varphi) \geqslant 0$ for all φ, and the assumption that the separatrices Γ_2', Γ_2'' do not intersect for small $\mu > 0$ is relaxed (if $\min_\varphi J_1(\varphi) = 0$, then the separatrices Γ_1', Γ_1'' may also intersect, but this effect is not captured by first-order perturbation theory). By the remark in Sect.3, an intersection of separatrices Γ_k'' consists of at least two homoclinic solutions. It turns out that the space

$$G = \{(c, \Lambda, l_1, l_2) \in \mathbf{R}^4: c^2 \geqslant l_1 > 0,\ l_2 > 0,\ \Lambda > 0\}$$

contains a closed set M_0 with no interior points, satisfying the following condition: if

$$(c, \Lambda, l_1, l_2) \in G \setminus M_0$$
$$l_1 = \alpha_1^2 + \beta_1^2, \quad l_2 = \alpha_2^2 + \beta_2^2$$

then for any sufficiently small $\mu > 0$ the separatrices Γ_k'' have at least one transversal intersection (homoclinic solution). Using the form of the expressions for $J_i(\varphi)$ /1/ and Alekseyev's theorem, we obtain the desired result.

Theorem 7. The parameter space pr of the perturbed Euler-Poinsot problem contains a closed set M, with no interior points, such that if $\mathrm{pr} \notin M$ and $\mu < \mu(\mathrm{pr})$ the problem has quasirandom solutions. The sets M_0, M have measure zero.

The idea of the proof is based, first, on the fact that near Γ_2 the separatrices Γ_1', Γ_2' are regular analytic surfaces which, as surfaces defined in the space

$$\mathbf{R}^1\{s = \Lambda\mu^{-1}\omega\} \times \{\varphi_2 \in (-C_6, C_6)\} \times S^1\{\varphi\} \tag{6.2}$$

are $O(\mu \ln \mu)$ -close in the C^ω-metric to the surfaces

$$s = J_1(\psi_1), \quad \psi_2 = \psi_1 - t - \Lambda^{-1} \ln J_1(\psi_1) \tag{6.3}$$
$$s = -J_2(\psi_2); \quad \psi_2 = \varphi - \varphi_2$$

and, second, on the following sharpened version of the remark in Sect.3: the separatrices Γ_k'' have contact of even order for at least two distinct homoclinic solutions. Assuming that for arbitrary small $\mu > 0$ Γ_k'' may have contact of order greater than one and letting $\mu \to 0$, one sees that the conditions $(c, \Lambda, l_1, l_2) \in M_0$ and $l_1 < c^2$ imply the existence of α_i, β_i such that $\alpha_i^2 + \beta_i^2 = l_i$ and

$$\alpha_1 = \alpha_2 = \alpha, \quad \beta_2 = (u - \beta_1)^{-1} u\beta_1 \tag{6.4}$$

$$\beta_1 [\alpha\beta_1^2 - 3u\alpha\beta_1 + 2\alpha u^2 + \Lambda u (c\alpha + l_1)] = 0 \tag{6.5}$$

where $u = \Lambda(c + \alpha) \neq 0$. It follows from (6.4), (6.5) that for each triple (Λ, c, l_1) there are at most seven values of l_2 such that $(c, \Lambda, l_1, l_2) \in M_0$. The set M_0 is a subset of the set of roots of a certain polynomial.

7. *Birth-death bifurcations of homoclinic and periodic solutions in the dynamics of a rigid body.*

Theorem 8. Fix $i = 1, 2, 3, 4$; $j = (i + 1) \bmod 4$. There exists a domain S_5 in the parameter space of the perturbed Euler-Poinsot problem such that for $\mathrm{pr} \in S_5$ and $\mu > 0$:

1) As $\mu \to 0$ all homoclinic solutions that pass once through the neighbourhood of two unperturbed separatrices x_i^*, x_j^* experience infinitely many birth-death bifurcations.

2) There exists a constant C_7 such that as $\mu \to 0$ all periodic solutions γ_N of the reduced system (1.1) which pass once in a period $2\pi N > C_7 - 2\Lambda^{-1} \ln \mu$ through the neighbourhood of two unperturbed separatrices x_i^*, x_j^* experience infinitely many birth-death bifurcations.

At the same time, every such homoclinic or periodic solution exists as long as $\ln \mu$ remains in a certain interval, whose length is less than a finite constant C_8 depending on pr.

3) The domain S_5 may be so chosen that for $\mu > 0$, $\mu \to 0$, the numbers of all homoclinic and periodic solutions considered in parts 1 and 2 take two distinct positive values infinitely many times.

Outline of proof. It will suffice to consider the problem studied in Sect.6. Fix $\Lambda > 0$, $c > 0$ and choose numbers $\alpha, \beta_1 \neq 0, \beta_2 = \beta_2^\circ \neq 0$ such that $l_1 = \alpha^2 + \beta_1^2 < c^2$, equalities (6.4) hold but (6.5) does not. Determine $l_2 = \alpha^2 + \beta_2^2$. Now let the arbitrary numbers α_i, β_i occurring in formulae (6.1) for $J_i(\varphi)$ satisfy the conditions $\alpha_i^2 + \beta_i^2 = l_i$, where l_i are the numbers just evaluated. Then for some values of $\psi_1 = \psi_1^\circ, \psi_2 = \psi_2^\circ, t = t_0$ we have

$$J_1(\psi_1) = -J_2(\psi_2) = c + \alpha \tag{7.1}$$
$$dJ_1(\psi_1)/d\psi_2 = -dJ_2(\psi_2)/d\psi_2 = -\beta_2^\circ \neq 0$$
$$d^2J_1(\psi_1)/d\psi_2^2 \neq -d^2J_2(\psi_2)/d\psi_2^2$$
$$(\psi_2 = \psi_1 - t - \Lambda^{-1} \ln J_1(\psi_1))$$

Conditions (7.1) are sufficient to ensure that $\mathrm{pr} \in S_5$. We now modify the numbers α_i, β_i slightly, and then, using the implicit function theorem, establish that conditions (7.1) are again satisfied for certain values of ψ_1, ψ_2, t close to $\psi_1^\circ, \psi_2^\circ, t_0$ respectively. Thus it may be assumed that the set S_5 is in fact a domain.

Let ξ', η', φ be normal coordinates near the solution $x = x_1(\varphi) = x_3(\varphi)$, $J' = \mu^{-1}\xi'\eta'$. With the coordinate ξ' in V_1 we associate a phase φ_2' by a formula of type (1.5), and with η' in V_2 a phase φ_1' by a formula of type (1.4).

The proof that conditions (7.1) are sufficient relies, first, on the fact that as μ goes through values corresponding to $t = t_0 \bmod 2\pi$ there occurs a birth or death bifurcation of two transversal intersection curves of the surfaces (6.3); second, one uses the $O(\mu \ln \mu)$-closeness of Γ_1'', Γ_2'' to the surfaces (6.3) in the space (6.2); and, third, the fact that the solution γ_N is in a domain $0 < J' < C_9$, where $C_9 = C_9(C_7) \to 0$ as $C_7 \to \infty$; the following assertion is also needed.

Lemma. Suppose that for some $t = t_0$ the curves (6.3) defined on $\mathbf{R}^1\{s\} \times S^1\{\psi_2\}$ have a transversal intersection. Then for sufficiently small $\mu > 0$, corresponding to $t = t_0 \bmod 2\pi$, and sufficiently large C_7 periodic solutions γ_N exist with the required properties in the neighbourhood of a suitable transversal homoclinic solution.

It is convenient to deal with this problem in terms of the succession mapping for system (1.1), defined in the plane $\varphi = \text{const}$. Let $q = x_1(\varphi)$, and let r be a transversal point of intersection in V_2 of the curves cut out by the separatrices Γ_k'' in the plane $\varphi = \text{const}$. The existence of the solutions γ_N is guaranteed by the method of symbolic dynamics /12-14/. By /13, 14/, to that end one must choose neighbourhoods V_q, V_r of the points q, r possessing certain properties (in other words, one must construct a suitable marching scheme /14/). In the neighbourhood of r we have coordinates $J_2' = J', \varphi_2'$, and also coordinates $J_1' = J', \varphi_1'$ obtained by continuation along the separatrix Γ_1''. It turns out that a suitable choice of V_q, V_r is the pair of curvilinear parallelograms defined by the conditions $|\xi'| < \sqrt{\mu}\rho$, $|\eta'| < \sqrt{\mu}\rho$ and $|J_1'| < C_{10}$, $|J_2'| < C_{10}$, respectively, where $|\ln \rho - C_{11}| < \pi\Lambda$ for small $\mu > 0$ (Fig.4).

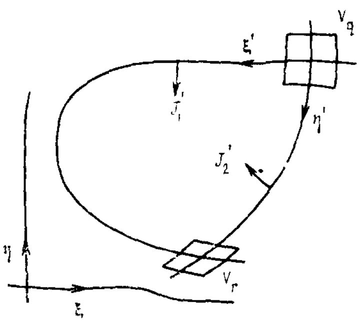

Fig.4

The method described here for constructing a marching scheme /13, 14/, based on using normal coordinates, is always applicable when the existence of transversally intersecting separatrices follows from first-order perturbation theory (see examples in /6/).

The proof of part 2 of the theorem relies essentially on the fact that every periodic solution must pass near some transversal homoclinic solution, and therefore is born and dies together with the latter. It is noteworthy that a similar phenomenon of the birth of periodic solutions close to transversal heteroclinic solutions has been observed in a numerical context /15/.

In order to prove part 3 of the theorem, it suffices to establish that the domain S_5 can be contracted in such a way that for any $t \in S^1$ the surfaces (6.3) are in contact along at most one curve $s = \text{const}$, $\psi_2 = \text{const}$, the contact is of first order, and the tangent plane is not parallel to the ψ_2 axis (see conditions (7.1)). It can be shown that these conditions are satisfied if α, β_1 are chosen as the numbers $\alpha = \sqrt{l_1}A$, $\beta_1 = \sqrt{l_1}B$, where $A^2 + B^2 = 1, B \neq 0$, with B, l_1 sufficiently small ($|B| < B_0, l_1 < l_1(B)$). In that situation the surfaces (6.3) are in contact for only two distinct values of $t \in S^1$.

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PLANAR STANDING AND MARKING-TIME REGIMES OF A BIPEDAL WALKING DEVICE*

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A walking device standing on one leg, not fastened at its points of support, is considered. A study is made of how the device maintains equilibrium of its supporting leg by compensating oscillations of its body. Phase trajectories are analysed. Conditions are investigated under which one-way communication is maintained and discontinued while the device is moving. Marking-time regimes are constructed.

The problem of a standing walking device is interesting, first, as a problem in the dynamics of servosystems, and, second, as a limiting case of the problem of locomotion. Marking-time regimes may be used in constructing a model of space locomotion**. (**Beletskii V.V. and Golubitskaya M.D., Model problem of the dynamics of bipedal space locomotion. Preprint 194, Moscow, Inst. Prikl. Mat. Akad. Nauk SSSR, 1982).

1. ***Description of the model. Equations of the standing problem.*** We consider a bipedal walking device consisting of a heavy rigid body and a pair of identical weightless legs (Fig.1); each leg may consist of one or several segments. The legs are attached to the body of the device by double hinges at a point O. The device is assumed to be supported on one leg only. The leg is in contact with the supporting surface at a single point S, at which there acts a reaction force $\mathbf{R}_S$; communication with the surface is one-way (non-restoring). At the hinge O a controlling torque $\mathbf{Q}$ acts on the body and a torque $-\mathbf{Q}$ on the leg.

We assume that the supporting leg is maintained in equilibrium - the suspension point O and support point S remain fixed. The system is subject to feedback: the motion of the body is designed to maintain equilibrium of the supporting leg.

We shall consider planar regimes of motion. Fix a coordinate frame $NXYZ$ (Fig.1), where N is the origin and the NZ axis is directed vertically upward. The support and suspension points are assumed to lie in the NYZ plane: $S = (0, d, 0)$, where $d = \text{const}, d > 0$, is the horizontal displacement of the support, and $O = (0, 0, H)$, where $H = \text{const}, H > 0$, is the height of the suspension point of the legs. It is assumed that the body does not spin; the centre of mass C moves in the NYZ plane.

We adopt the following notation: θ is the angle between the NZ axis and the vector $\mathbf{OC}$ in the positively oriented system $NXYZ$ (Fig.1), t is the time, g is the acceleration of free

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